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Uniqueness of Solutions of Differential Equations

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UNIQUENESS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Uniqueness of solutions for ordinary differential equations is studied. The classical theorems which guarantee uniqueness are surveyed, including discussion and examples. Other results concerning uniqueness are considered in the final chapter, including the relationship between convergence of successive approximations and uniqueness, non-uniqueness and continuous dependence on initial conditions.

Table of Contents

Chapter 1

INTRODUCTION

Uniqueness deals with having only one solution to an initial value problem. Nonuniqueness deals with more than one solution. Uniqueness in and of itself is an important concept to understand, especially within our technological world. For instance, some software packages, such as Phaser, would yield only one solution to the problem $\frac{dy}{dx} = y^{1/2}$, where $y(0) = 0$. One solution is incorrect because there is more than one solution to this problem ($y \equiv 0$ and $y = \frac{x^2}{4}$). It is also important to realize that in applications there is often only one valid solution because of real-life constraints. However, the initial value problem used to model the problem could have more than one solution.

Let's first look at uniqueness and non-uniqueness through a geometric perspective. A technique that is useful in graphing solutions to a differential equation is to sketch the direction field for the equation. In order to describe the method further, we note that a first order equation $\frac{dy}{dx} = f(x, y)$ gives the slope of the tangent line that a solution to the equation must have at each point. By plotting the directions of the various points in the plane, the direction field for the equation is constructed. A direction field shows uniqueness when there is only one choice of path to follow through the direction field for a given initial point. A direction field shows non-uniqueness when there is more than one choice of path to follow.

Example I: Consider the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= y^{\frac{1}{2}}, \\ y(0) &= 0. \end{aligned}$$

In Figure I, the direction field is given. Note that this initial value problem has more than one solution, since there is more than one choice of direction from $(0, 0)$.

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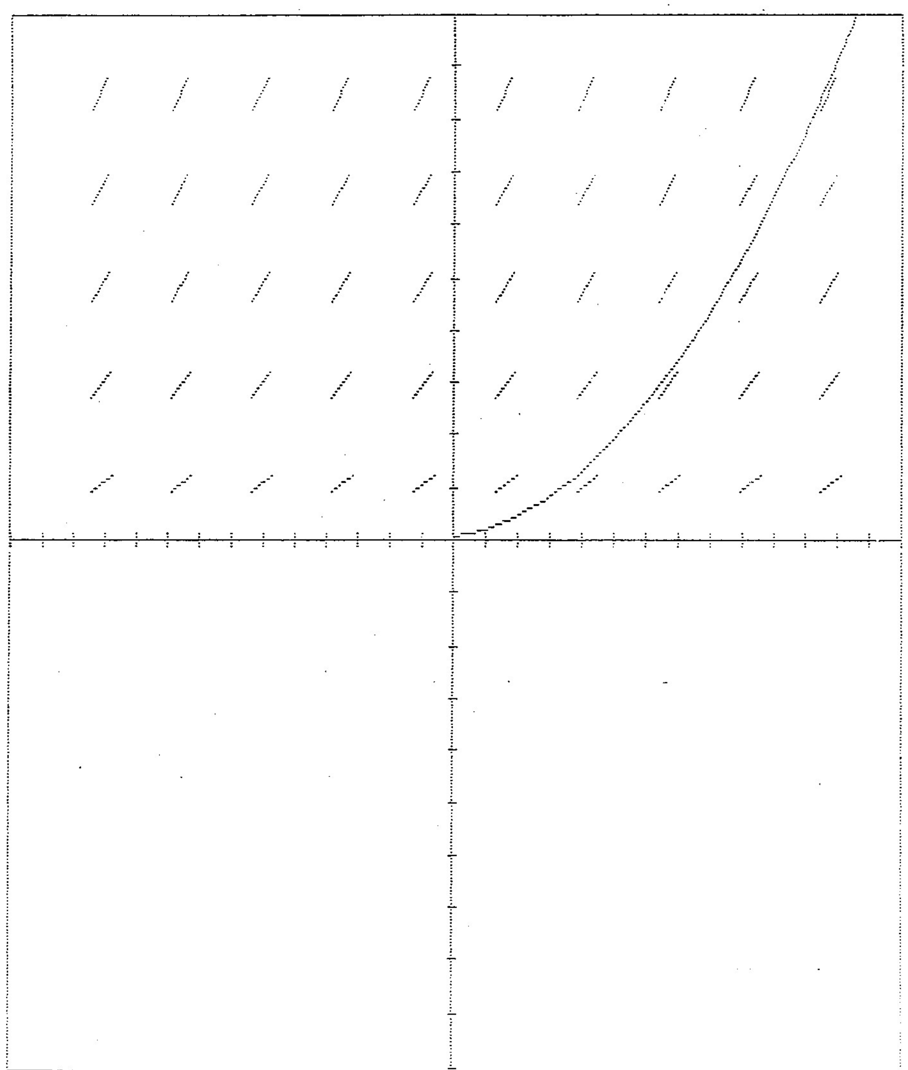
Figure I



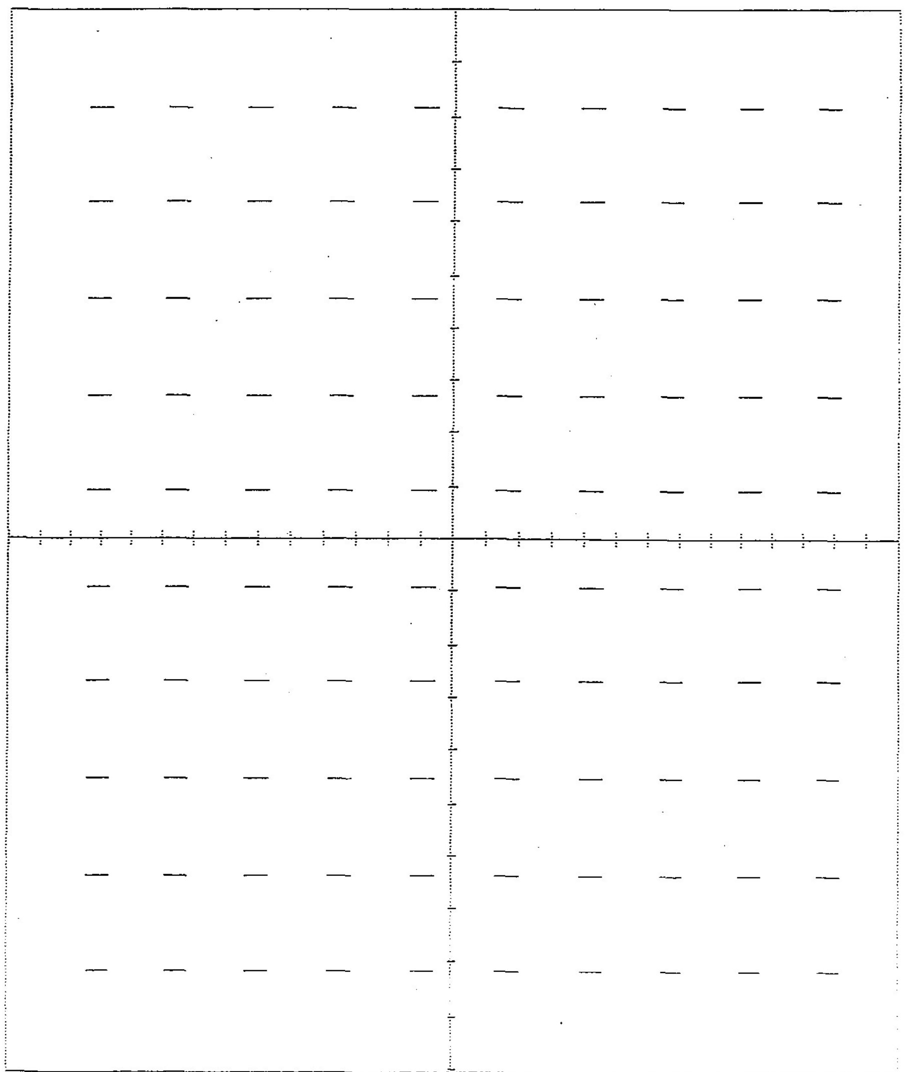
Direction field for example I.

Example II: Now consider

$$\begin{aligned}\frac{dy}{dx} &= 0,\\ y(0) &= 0.\end{aligned}$$

The direction field is given in Figure II. Note that in this example, there is only one choice of direction from (0,0).

Figure II



Direction field for example II.

Chapter 2

THEOREMS WHICH GUARANTEE UNIQUENESS

We now survey some of the known theorems which specify requirements that guarantee uniqueness of solutions. We note that if f is continuous, existence of solutions is guaranteed by the well-known existence theorem of Peano [8]. (In fact, the existence of both a maximal and minimal solution is guaranteed.) Part of the presentation in this section is based on the one in [5].

Theorem 1 [10]:
Given the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$, assume that f and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle $R = \{(x,y)$: $x_0 \leq x \leq x_0 + a$ and $|y - y_0| \leq b\}$, where $a, b > 0$. Then, the initial value problem has a unique solution in an interval $x_0 \leq x \leq x_0 + h$, where h is some positive number.

Before we prove Theorem 1, we shall note the following:

Lemma A (Gronwall's Lemma):
Let $r : \mathbf{R} \to \mathbf{R}$ be continuous and non-negative. Let $\sigma, L \geq 0$ and $x_0 \in \mathbf{R}$ be given. Suppose also that $r(x) \leq \sigma + \int_{x_0}^{x} Lr(s)ds$ for $x \geq x_0$. Then, $r(x) \leq \sigma \exp[L(x - x_0)]$, for $x \geq x_0$.

Proof of Lemma A:
Let $B(x) = \int_{x_0}^{x} Lr(s)ds$, for $x \geq x_0$. From the hypotheses, we have $Lr(x) \leq L\sigma + LB(x)$. Furthermore, we have $\frac{dB(x)}{dx} = Lr(x) \leq L\sigma + LB(x)$ and so $\frac{dB(x)}{dx} - LB(x) \leq L\sigma$. Hence,

$$\tfrac{dB(s)}{ds}\exp[L(x_0-s)] - LB(s)\exp[L(x_0-s)] \le L\sigma\exp[L(x_0-s)]$$

$$\tfrac{d}{ds}(\exp[L(x_0-s)]B(s)) \le L\sigma\exp[L(x_0-s)]$$

$$\textstyle\int_{x_0}^{x} d(\exp[L(x_0-s)]B(s)) \le \int_{x_0}^{x} L\sigma\exp[L(x_0-s)]ds$$

$$\exp L(x_0-s)]B(s)]_{x_0}^{x} \le L\sigma\tfrac{\exp[L(x_0-s)]}{-L}]_{x_0}^{x}$$

$$\exp[L(x_0-x)]B(x) - B(x_0) \le -\sigma\exp[L(x_0-x)] + \sigma$$

$$B(x) \le -\sigma + \sigma\exp[L(x-x_0)].$$

Now,

$$r(x) \le \sigma + B(x)$$

$$r(x) - \sigma \le B(x) \le -\sigma + \sigma\exp[L(x-x_0)]$$

$$r(x) \le \sigma\exp[L(x-x_0)].$$

∎

Proof of Theorem 1:

Claim: There exists an $L > 0$ such that $\mid f(x,y) - f(x,z) \mid\le L \mid y - z \mid$ for any $(x,y),(x,z) \in R$, i.e., f is Lipschitz continuous in its second variable.

Proof of Claim:

Since f and $\dfrac{\partial f}{\partial y}$ are continuous and R is closed and bounded, then f and $\dfrac{\partial f}{\partial y}$ are bounded on R, that is, there exists an M and L such that for all $(x,y) \in R$, $\mid f(x,y) \mid\le M$ and $\mid \dfrac{df}{dy}(x,y) \mid\le L$. Choose $(x,y),(x,z) \in R$ with $y < z$. By the mean value theorem, there exists a $y^* \in [y,z]$ such that $\dfrac{\partial f}{\partial y}(x,y^*) = \dfrac{f(x,y)-f(x,z)}{y-z}$. Thus, $\mid \dfrac{\partial f}{\partial y}(x,y^*) \mid= \dfrac{\mid f(x,y)-f(x,z)\mid}{\mid y-z\mid}$, and so $\mid f(x,y) - f(x,z) \mid=\mid \dfrac{\partial f}{\partial y}(x,y^*) \mid \mid y-z \mid\le L \mid y-z \mid$, establishing the claim.

Suppose there are two solutions u_1 and u_2 on $x_0 \le x \le x_0 + h$ for some $h > 0$. We then have $u_1(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s,u_1(s))ds$ and $u_2(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s,u_2(s))ds$.

Now, we have the following for $x_0 \leq x \leq x_0 + h$:

$$0 \leq | u_1(x) - u_2(x) | \tag{2.1}$$

and

$$\begin{aligned} &= | y_0 + \int_{x_0}^{x} f(s, u_1(s))ds - [y_0 + \int_{x_0}^{x} f(s, u_2(s))ds] | \\ &= | \int_{x_0}^{x} [f(s, u_1(s)) - f(s, u_2(s))]ds | \\ &\leq \int_{x_0}^{x} | f(s, u_1(s)) - f(s, u_2(s)) | ds \\ &\leq \int_{x_0}^{x} L | u_1(s) - u_2(s) | ds, \text{ by the claim.} \end{aligned}$$

We now apply Gronwall's Lemma, letting $r(s) =| u_1(s) - u_2(s) |$ and $\sigma = 0$, thus,

$$| u_1(x) - u_2(x) | \leq 0. \tag{2.2}$$

From (2.1) and (2.2) we have $0 \leq | u_1(x) - u_2(x) | \leq 0$ which implies that $| u_1(x) - u_2(x) | = 0$. At this point, we can conclude that $u_1(x) = u_2(x)$ on $x_0 \leq x < x_0 + h$. ∎

Theorem 2 [7]:
Given the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, assume that f is a continuous function on R and f is Lipschitz in its second variable: For all $(x, y), (x, z) \in R$, $| f(x, y) - f(x, z) | \leq L | y - z |$ for some $L > 0$, where $R = \{(x, y) : x_0 \leq x \leq x_0 + a$, and $| y - y_0 | \leq b\}$ where $a, b > 0$. Then, the initial value problem has a unique solution in an interval $x_0 \leq x \leq x_0 + h$, where h is some positive number.

The proof of Theorem 2 is exactly the same as the proof of Theorem 1, except the claim is assumed here. Thus, Theorem 1 is a special case of Theorem 2.

The following is an example to which Theorem 1 does not apply. However, Theorem 2 does apply. Consider $f(x, y) =| y |, y(0) = 0$. Let $R = \{(x, y) : 0 \leq x \leq 1, -1 \leq y \leq 1\}$. We have

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} -1 & 0 \leq x \leq 1, -1 \leq y < 0 \\ 1 & 0 \leq x \leq 1, 0 < y \leq 1 \\ \text{undefined} & 0 \leq x \leq 1, y = 0. \end{cases}$$

The $\dfrac{\partial f}{\partial y}$ does not exist on all of R, which means that Theorem 1 does not apply. However, for any $(x,y),(x,z) \in R, |f(x,y) - f(x,z)| = || y | - | z || \leq |y-z|$, which implies f is Lipschitz with constant of 1, implying that Theorem 2 does apply.

Theorem 3 [9]:
Assume that the function $g(x,y)$ is continuous and nonnegative in a rectangle $R_1 = \{(x,y) : x_0 \leq x \leq x_0 + a, 0 \leq y \leq 2b\}$ where $a, b > 0$ and for every $x_1 \in (x_0, x_0 + a], y(x) \equiv 0$ is the only differentiable function on $x_0 \leq x \leq x_1$ which satisfies $y' = g(x,y), y(x_0) = 0$. Also assume that f is a continuous function on R and for all $(x,y),(x,z) \in R, |f(x,y) - f(x,z)| \leq g(x, | y - z |)$,where $R = \{(x,y) : x_0 \leq x \leq x_0 + a$, and $|y - y_0| \leq b\}$. Then, the initial value problem $y' = f(x,y), y(x_0) = y_0$ has a unique solution in an interval $x_0 \leq x \leq x_0 + h$, where h is some positive number.

We shall need a few lemmas before proving Theorem 3. In the next lemma, we shall make use of the Dini derivatives. These are defined as follows:

$$
\begin{aligned}
D^+u(t) &= \limsup_{h \to 0^+} \frac{u(t+h) - u(t)}{h} \\
D_+u(t) &= \liminf_{h \to 0^+} \frac{u(t+h) - u(t)}{h} \\
D^-u(t) &= \limsup_{h \to 0^-} \frac{u(t+h) - u(t)}{h} \\
D_-u(t) &= \liminf_{h \to 0^-} \frac{u(t+h) - u(t)}{h}.
\end{aligned}
$$

If $D^+u(t) = D_+u(t)$, the right derivative exists and likewise if $D^-u(t) = D_-u(t)$, then the left derivative exists. If all four Dini derivatives agree, then the function is differentiable. The proofs of the next three lemmas can be found in [5].

Lemma B:
Let $v, w \in C[[x_0, x_0+a], \mathbf{R}]$, where $a > 0$ and assume $Du(x) \leq w(x)$ for $x \in [x_0, x_0+a]$ except for at most a countable subset, where D denotes any Dini derivative. Then, $D_-v(x) \leq w(x)$ for $x \in [x_0, x_0 + a]$.

Lemma C:
Let $g \in C[E, \mathbf{R}]$, where E is an open subset of $\mathbf{R}^2$ and let $(x_0, y_0) \in E$. Let $[x_0, x_0 + a]$ be an interval of existence of the maximal solution r *of*

$$
\begin{aligned}
y' &= g(x,y), \\
y(x_0) &= y_0.
\end{aligned}
$$

Let $x_1 \in (x_0, x_0 + a]$. Then, there exists an $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, the maximal solution $r(x, \epsilon)$ of

$$y' = g(x, y) + \epsilon,\ y(x_0) = y_0 + \epsilon$$

exists on $[x_0, x_1]$ and $\lim_{\epsilon \to 0^+} r(x, \epsilon) = r(x)$ uniformly on $[x_0, x_1]$.

Lemma D:

Let ϵ be on open subset of $\mathbf{R}^2$ and let $g \in C[E, \mathbf{R}]$. Assume that $v, w \in C[[x_0, x_0 + a], \mathbf{R}]$ for $a > 0$ and $(x, v(x)), (x, w(x)) \in E$ for $x \in [x_0, x_0 + a]$. Suppose further that $v(x_0) < w(x_0)$, and for $x \in [x_0, x_0 + a]$,

$$\begin{aligned} D_{-}v(x) &\leq g(x, v(x)) \\ D_{-}w(x) &> g(x, w(x)) \end{aligned}$$

Then, $v(x) < w(x)$ for $x \in [x_0, x_0 + a]$.

Lemma E [5]:

Let E be an open set in $\mathbf{R}^2$ and $g \in C[E, \mathbf{R}]$. Suppose that $[x_0, x_0 + a]$ is an interval in which the maximum solution $r(x)$ of $y' = g(x, y)$, $y(x_0) = y_0$ exists. Let $m \in C[[x_0, x_0 + a], \mathbf{R}], (x, m(x)) \in E$ for $x \in [x_0, x_0 + a], m(x_0) \leq y_0$, and for a fixed Dini derivative D, $Dm(x) \leq g(x, m(x))$ except possibly on a countable set. Then, $m(x) \leq r(x)$, where $x \in [x_0, x_0 + a]$.

Proof of Lemma E:
Using Lemma B, we have

$$D_{-}m(x) \leq g(x, m(x)),\ \text{where } x \in [x_0, x_0 + a]. \tag{2.3}$$

Now, let $x_1 \in (x_0, x_0 + a]$. By Lemma C, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the maximal solution $r(x, \epsilon)$ of

$$y' = g(x, y) + \epsilon, y(x_0) = y_0 + \epsilon$$

exists over $[x_0, x_1]$ and $\lim_{\epsilon \to 0} r(x, \epsilon) = r(x)$ uniformly on $[x_0, x_1]$. Using (2.3), and the fact that $m(x_0) \leq y_0 < y_0 + \epsilon = r(x_0, \epsilon)$, we have by Lemma D, $m(x) < r(x, \epsilon)$. Therefore, $m(x) \leq \lim_{\epsilon \to 0} r(x, \epsilon) = r(x)$. ∎

Proof of Theorem 3:

Suppose there are two solutions u_1 and u_2 on $x_0 \le x \le x_0 + h$ for some $h > 0$. Define m by $m(x) = |u_1(x) - u_2(x)|$. We then have

$$\begin{aligned}
D^+m(x) &= \limsup_{x_1 \to x^+} \frac{m(x_1) - m(x)}{x_1 - x} \\
&= \limsup_{x_1 \to x^+} \frac{|\, u_1(x_1) - u_2(x_1) \,| - |\, u_1(x) - u_2(x) \,|}{x_1 - x} \\
&\le \limsup_{x_1 \to x^+} \left| \frac{u_1(x_1) - u_2(x_1) - (u_1(x) - u_2(x))}{x_1 - x} \right| \\
&= \left| \limsup_{x_1 \to x^+} \left(\frac{u_1(x_1) - u_1(x)}{x_1 - x} - \frac{u_2(x_1) - u_2(x)}{x_1 - x} \right) \right| \\
&= \left| \lim_{x_1 \to x^+} \frac{u_1(x_1) - u_1(x)}{x_1 - x} - \lim_{x_1 \to x^+} \frac{u_2(x_1) - u_2(x)}{x_1 - x} \right| \\
&= |\, u_1'(x) - u_2'(x) \,| \\
&= |\, f(x, u_1(x)) - f(x, u_2(x)) \,| \\
&\le g(x, |\, u_1(x) - u_2(x) \,|) \\
&= g(x, m(x)).
\end{aligned}$$

By Lemma E, $m(x) \le r(x)$, where r is the maximal solution of $y' = g(x, y), y(x_0) = 0$. However, by assumption, $r = 0$. We then have $0 \le |u_1(x) - u_2(x)| = m(x) \le r(x) = 0$ which implies $|u_1(x) - u_2(x)| = 0$ and hence $u_1(x) = u_2(x)$. ∎

We now note that Theorem 2 is a special case of Theorem 3. Assume that the hypotheses of Theorem 2 are satisfied. From this, we know there exists an $L > 0$, such that $|\, f(x, w) - f(x, z) \,| \le L \,|\, w - z \,|$ for all $(x, w), (x, z) \in R$. Let $g(x, y) = Ly$. Note that g is continuous on $\{(x, y) : x_0 \le x \le x_0 + a, 0 \le y \le 2b\}$. Now consider the initial value problem $y' = Ly, \quad y(x_0) = 0$. Note that $y \equiv 0$ is a solution because $y' = 0$, $Ly = L(0) = 0$ and $y(x_0) = 0$. We have that $g(x, y) = Ly$ is continuous and $\dfrac{\partial g}{\partial y} = L$ is continuous. Hence, by Theorem 1 $y \equiv 0$ is the only solution to $y' = Ly, \ y(x_0) = 0$. Thus, f and g satisfy the hypotheses of Theorem 3.

Theorem 4 [4]:

Assume that the function $g(x, y)$ is continuous and nonnegative in a rectangle $R_1 = \{(x, y) : x_0 \le x \le x_0 + a, 0 \le y \le 2b\}$, where $a, b > 0$, and for every $x_1 \in (x_0, x_0 +$

$a]$ $y(x) \equiv 0$ is the only differentiable function on $x_0 \leq x \leq x_1$ for which

$$\begin{aligned}
y'_+(x_0) &\equiv \lim_{x \to x_0^+} \frac{y(x) - y(x_0)}{x - x_0} \text{ exists,} \\
y'(x) &= g(x, y(x)), x_0 \leq x \leq x_1, \\
y(x_0) &= y'_+(x_0) = 0.
\end{aligned}$$

Assume also that f is a continuous function on R and $|f(x,y) - f(x,z)| \leq g(x, | y - z |)$ holds for $(x,y), (x,z) \in R$ where $x \neq x_0$, for $R = \{(x,y) : x_0 \leq x \leq x_0 + a, | y - y_0 | \leq b\}$. Then, the initial value problem has a unique solution in an interval $x_0 \leq x \leq x_0 + h$, where h is some positive number.

Lemma F:
Let E be an open subset of $\mathbf{R}^2$ and $g \in C[E, \mathbf{R}]$. Suppose that $m \in C[[x_0 - a, x_0], \mathbf{R}]$ for $a > 0, (x, m(x)) \in E$ for $x \in [x_0 - a, x_0], m(x_0) \geq y_0$ and for any fixed Dini derivative D,

$$Dm(x) \leq g(x, m(x)), x \in [x_0 - a, x_0]$$

Then, $m(x) \geq p(x)$ for all x as far as $p(x)$ exists to the left of x_0, where p is the minimal solution of

$$\begin{aligned}
y' &= g(x, y), \\
y(x_0) &= y_0.
\end{aligned}$$

The proof can be found in [5].

Lemma G [5]:
Let the function $g(x,y)$ satisfy the hypotheses of Theorem 4. Assume that the function $g_1(x,y)$ is continuous and nonnegative for $x_0 \leq x \leq x_0 + a$, $0 \leq y \leq 2b$, $g_1(x,0) = 0$, and $g_1(x,y) \leq g(x,y)$, $x \neq x_0$. Then, for every $x_1 \in (x_0, x_0+a]$, $y(x) = 0$ is the only differentiable function on $x_0 \leq x \leq x_1$, which satisfies $y' = g_1(x,y)$, $y(x_0) = 0$ for $x_0 \leq x \leq x_1$.

Proof of Lemma G:
We shall show that the maximal solution $r(x)$ of $y' = g_1(x,y)$, $y(x_0) = 0$ is identically zero. Suppose, on the contrary, that there exists a u with $x_0 < u < x_0 + a$, such that $r(u) > 0$. Because for $x \in (x_0, u]$ $g_1(x,y) \leq g(x,y)$ we have $r'(x) = g_1(x, r(x)) \leq g(x, r(x))$. Thus we have $r'(x) \leq g(x, r(x))$, $x_0 < x \leq u$. If $p(x)$ is the minimal solution of $y' = g(x,y), y(u) = r(u)$, an application of Lemma F shows that $p(x) \leq r(x)$, as far as $p(x)$ exists to the left of u. The solution $p(x)$ can be continued to $x = x_0$. If $p(T) = 0$ for some T satisfying $x_0 < T < u$ we can redefine p by $p(x) = 0$ for $x_0 \leq x \leq T$. Furthermore, since $g_1(x,y)$ is continuous at $(x_0, 0)$ and $g_1(x_0, 0) = 0$, $r'_+(x_0)$ exists and is equal to 0. This and the fact that $0 \leq p(x) \leq r(x)$ for $x \in [x_0, u]$ implies $p'_+(x_0) = 0$. But, we have assumed that $g(x,y)$ satisfies the hypotheses of Theorem 4. Hence, $p(u) = r(u) > 0$, a contradiction. Therefore, $r(x) = 0$. ∎

Proof of Theorem 4:
Define g_1 by $g_1(x,y) = \sup_{|w-z|=y} |f(x,w) - f(x,z)|$ for $x_0 \leq x \leq x_0 + a, 0 \leq y \leq 2b$. Note that for each x, y

$$g_1(x,y) = \sup_{|w-z|=y} |f(x,w) - f(x,z)| \leq \sup_{|w-z|=y} g(x, | w - z |) = g(x,y),$$

so $g_1(x,y)$ is finite. We shall show that g_1 satisfies the requirements on g in Lemma G. Since $f(x,y)$ is continuous on $x_0 \leq x \leq x_0 + a, | y - y_0 | \leq b$, then $g_1(x,y)$ is continuous on $x_0 \leq x \leq x_0 + a, 0 \leq y \leq 2b$. Also, from the definition of g_1, we clearly have that g_1 is nonnegative on $x_0 \leq x \leq x_0 + a, 0 \leq y \leq 2b$. Note also that $g_1(x,0) = \sup_{|w-z|=0} |f(x,w) - f(x,z)| = |f(x,w) - f(x,w)| = 0$. Thus, Lemma G applies to g_1 and we can conclude that for every $x_1 \in [x_0, x_0+a]$, $y(x) \equiv 0$ is the only differentiable function on $x_0 \leq x \leq x_1$ which satisfies $y' = g_1(x,y), y(x_0) = 0$.

Also, for each $(x,y), (x,v) \in R$ $g_1(x, | y - v |) = \sup_{|w-z|=|y-v|} |f(x,w) - f(x,z)| \geq |f(x,y) - f(x,v)|$. We may now apply Theorem 3. ∎

We note that Theorem 3 is a special case of Theorem 4. In Theorem 3, $y \equiv 0$ is the only function that satisfies a set of requirements. In Theorem 4, $y \equiv 0$ is the only function that satisfies that same set of requirements plus an extra requirement $y'_+(t_0) = 0$. Uniqueness of y is preserved since it is the only function to satisfy that first set of requirements and existence of y is retained since if $y \equiv 0$, then $y'_+(t_0) = 0$. The hypotheses on f of Theorem 4 would also apply if those in Theorem 3 were assumed.

Chapter 3

OTHER TOPICS CONCERNING UNIQUENESS

We shall first investigate the relationship between uniqueness and convergence of successive approximations. Given the initial value problem

$$\begin{aligned} y' &= f(x,y), \\ y(x_0) &= y_0, \end{aligned}$$

we define a sequence $y_0, y_1, y_2, \ldots y_n, \ldots$ of functions, known as successive approximations, as follows:

$$\begin{aligned} &y_0(x) = y_0 \\ &y_1(x) = y_0 + \textstyle\int_{x_0}^{x} f[t,\ y_0(t)]dt, \\ &y_2(x) = y_0 + \textstyle\int_{x_0}^{x} f[t,\ y_1(t)]dt, \\ &\qquad\vdots \\ &y_n(x) = y_0 + \textstyle\int_{x_0}^{x} f[t,\ y_{n-1}(t)]dt. \end{aligned}$$

Let us show how it works by means of a few examples.

Example III: Consider

$$\begin{aligned} y' &= x^2, \\ y(2) &= 1. \end{aligned}$$

We then have

$$\begin{aligned} &y_0(x) = 1 \\ &y_1(x) = 1 + \textstyle\int_{2}^{x} t^2 dt = \frac{x^3}{3} - \frac{5}{3}, \\ &y_2(x) = 1 + \textstyle\int_{2}^{x} t^2 dt = \frac{x^3}{3} - \frac{5}{3}, \\ &\qquad\vdots \\ &y_n(x) = 1 + \textstyle\int_{2}^{x} t^2 dt = \frac{x^3}{3} - \frac{5}{3}. \end{aligned}$$

Note that $\{y_n\}$ converges to $\frac{x^3}{3} - \frac{5}{3}$ which is a solution to the given initial value problem.

Example IV: Now consider

$$\begin{aligned} y' &= y, \\ y(0) &= 1. \end{aligned}$$

We then have

$$\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + \textstyle\int_0^x dt = 1 + x \\
y_2(x) &= 1 + \textstyle\int_0^x (1+t)dt = 1 + x + \frac{x^2}{2} \\
y_3(x) &= 1 + \textstyle\int_0^x (1 + t + \frac{t^2}{2})dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \\
&\vdots \\
y_n(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.
\end{aligned}$$

The successive approximations do in fact converge to a solution $y(x) = e^x$.

Theorem 5 [5]:
Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions on $R = [(x,y) : x_0 \leq x \leq x_0 + a$ and $| y - y_0 |\leq b\}$ where $a, b > 0$. Then there exists a number $h > 0$ with the property that the initial value problem $y' = f(x,y), y(x_0) = y_0$ has one and only one solution on $[x_0, x_0 + h]$.

Proof of Theorem 5:
We note that it suffices to consider the equation $y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt$ rather than the original initial value problem. Now set up the sequence of successive approximations as defined earlier. Next, we observe that $y_n(x)$ is the n^{th} partial sum of the series of functions.

$$\begin{aligned}
y_0(x) + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] = y_0(x) &+ [y_1(x) - y_0(x)] \\
&+ \cdots\cdots + [y_n(x) - y_{n-1}(x)] \\
&+ \cdots,
\end{aligned}$$

so the convergence of the sequence of successive approximations is equivalent to the convergence of this series.

Now, we choose h. Using the hypotheses of the theorem, we have that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on the rectangle R. Since R is closed and bounded, each of these functions is necessarily bounded on R. Thus, there exists constants M and

K such that $| f(x,y) | \leq M$ and $| \frac{\partial f}{\partial y}(x,y) | \leq K$ for all points (x,y) in R. If (x,y_1) and (x,y_2) are distinct points in R, then the mean value theorem guarantees that $| f(x,y_1) - f(x,y_2) | = | \frac{\partial f}{\partial y}(x,y^*) || y_1 - y_2 |$, for some number y^* between y_1 and y_2. So, $| f(x,y_1) - f(x,y_2) | \leq K | y_1 - y_2 |$, (i.e., f is Lipschitz in its second variable). Now, choose $h > 0$ such that $Kh < 1$ and the rectangle R', defined by $x_0 \leq x \leq x_0 + h$ and $| y - y_0 | \leq Mh$, is contained in R.

We first show there exists $y(x)$ such that $y_0(x) + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] = y(x)$. We need to show that the series

$$| y_0(x) | + | y_1(x) - y_0(x) | + | y_2(x) - y_1(x) | + \cdots + | y_n(x) - y_{n-1}(x) | + \cdots \quad (3.1)$$

converges. We estimate the terms $| y_n(x) - y_{n-1}(x) |$ to accomplish this. We know that $R' \subseteq R$. For the graph of y_n to remain in R' and thus in R, we need $| y_n(x) - y_0 | \leq Mh$ for all $x \in [x_0, x_0 + h]$. Thus,

$$\begin{aligned} | y_1(x) - y_0 | &= | y_0 + \int_{x_0}^{x} f(t, y_0(t))dt - y_0 | \\ &= | \int_{x_0}^{x} f(t, y_0(t))dt | \\ &\leq \int_{x_0}^{x} | f(t, y_0(t)) | dt \\ &\leq \int_{x_0}^{x} M dt \\ &= M[x - x_0] \leq Mh. \end{aligned}$$

The process works similarly for the other terms. We also know that $| y_1(x) - y_0 |$ is continuous. Hence, we can define a constant a by $a = \max_{[x_0, x_0+h]} | y_1(x) - y_0 |$ and thus $| y_1(x) - y_0(x) | \leq a$. Next, since $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R', we have $| f[t, y_1(t)] - f[t, y_0(t)] | \leq K | y_1(t) - y_0(t) | \leq Ka$. Next, we have

$$\begin{aligned} | y_2(x) - y_1(x) | &= | y_0 + \int_{x_0}^{x} f(t, y_1(t))dt - (y_0 + \int_{x_0}^{x} f(t, y_0(t))dt) | \\ &= | \int_{x_0}^{x} [f(t, y_1(t)) - f(t, y_0(t))]dt | \\ &\leq \int_{x_0}^{x} | f(t, y_1(t)) - f(t, y_0(t)) | dt \\ &\leq \int_{x_0}^{x} K | y_1(t) - y_0(t) | dt \\ &\leq \int_{x_0}^{x} Ka dt = Ka(x - x_0) \leq Kah = a(Kh)^1. \end{aligned}$$

Similarly, $| f[t, y_2(t)] - f[t, y_1(t)] | \leq K | y_2(t) - y_1(t) | \leq K^2ah$. Then, $| y_3(x) - y_2(x) | = | \int_{x_0}^{x} (f[t, y_2(t)] - f[t, y_1(t)])dt | \leq (K^2ah)h = a(Kh)^2$. We continue with this pattern and find that $| y_n(x) - y_{n-1}(x) | \leq a(Kh)^{n-1}$ for every $n = 1, 2, \cdots$. Each term of the series (3.1) is less than or equal to the corresponding term in the series

$$| y_0 | + a + a(Kh) + a(Kh)^2 + \cdots + \cdots + a(Kh)^{n-1} + \cdots. \tag{3.2}$$

The fact that $Kh < 1$ ensures us that (3.2) converges, and (3.1) converges by the comparison test. Call the limit of successive approximations $y(x)$.

Second, we need to show that y is a solution. By a well-known theorem in analysis, since y_n converges uniformly to y, y must be continuous. From this point, we must show

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

We know that $y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]dt = 0$ and thus, $y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]dt - y(x) + y_0 + \int_{x_0}^{x} f[t, y(t)]dt = -y(x) + y_0 + \int_{x_0}^{x} f[t, y(t))]dt$. Then, $y_n(x) - y(x) + \int_{x_0}^{x} [f(t, y(t)) - f(t, y_{n-1}(t))]dt = -y(x) + y_0 + \int_{x_0}^{x} f(t, y(t))dt$. Hence,

$$\begin{aligned}
| \textstyle\int_{x_0}^{x} [f(t, y_{n-1}(t)) - f(t, y(t))]dt | &\leq \\
\textstyle\int_{x_0}^{x} | f[t, y_{n-1}(t)] - f[t, y(t)] | \, dt &\leq \\
\textstyle\int_{x_0}^{x} K | y_{n-1}(t) - y(t) | \, dt &\leq \\
\textstyle\int_{x_0}^{x} K \max_{t \in [x_0, x]} | y_{n-1}(t) - y(t) | \, dt &= \\
(x - x_0) K \max_{t \in [x_0, x]} | y_{n-1}(t) - y(t) | &\leq \\
hK \max_{t \in [x_0, x]} | y_{n-1}(t) - y(t) | &.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&| y(x) - y_0 - \textstyle\int_{x_0}^{x} f(t, y(t))dt | = \\
&| y(x) - y_n(x) + \textstyle\int_{x_0}^{x} [f(t, y_{n-1}(t)) - f(t, y(t))]dt | \leq \\
&| y(x) - y_n(x) | + | \textstyle\int_{x_0}^{x} [f(t, y_{n-1}(t)) - f(t, y(t))]dt
\end{aligned}$$

which is less than or equal to

$$| y(x) - y_n(x) | + hK \max_{t \in [x_0, x]} | y_{n-1}(t) - y(t) | . \tag{3.3}$$

We can make (3.3) arbitrarily small since $\{y_n\}$ convergences uniformly to y. Hence,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

We may now apply Theorem 1 to obtain uniqueness of solutions. ∎

The hypotheses of Theorem 5 imply that successive approximations converge, and they also imply uniqueness. However, does uniqueness imply that successive approximations converge or vice versa? We shall now explore this question using an example from [2]. Let f be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0, y \in (-\infty, \infty) \\ 2x & \text{if } x \in (0,1], y \in (-\infty, 0) \\ 2x - \frac{4y}{x} & \text{if } x \in (0,1], y \in [0, x^2] \\ -2x & \text{if } x \in (0,1], y \in (x^2, \infty) \end{cases}$$

and consider $y' = f(x,y)$, $y(0) = 0$.

Proposition 1:
We shall first verify that this problem has a unique solution:
If $f(x,y)$ is non-increasing in y for each fixed x and $y' = f(x,y), y(0) = y_0$ has a solution, then that solution is unique.

Proof.
Let y_1, y_2 be solutions. Assume there exists $\overline{x}$ such that $y_1(\overline{x}) \neq y_2(\overline{x})$. Without loss of generality, we assume $y_1(\overline{x}) > y_2(\overline{x})$. Then since y_1, y_2 are continuous and $y_1(0) = y_2(0)$, there exists x_1 such that $y_1(x_1) = y_2(x_1)$ and $y_1(x) > y_2(x)$ for $x \in (x_1, \overline{x}]$. Let $x \in (x_1, \overline{x}]$.

$$\begin{aligned} y_2(x) &= y_2(0) + \textstyle\int_0^x f(s, y_2(s))ds \\ &= y_2(0) + \textstyle\int_0^{x_1} f(s, y_2(s))ds + \int_{x_1}^{x} f(s, y_2(s))ds \end{aligned}$$

Then,

$$\begin{aligned} y_2(x) &= y_2(x_1) + \textstyle\int_{x_1}^{x} f(s, y_2(s))ds \\ &\geq y_2(x_1) + \textstyle\int_{x_1}^{x} f(s, y_1(s))ds \\ &= y_1(x_1) + \textstyle\int_{x_1}^{x} f(s, y_1(s))ds \\ &= y_1(x). \end{aligned}$$

So, $y_2(x) \geq y_1(x)$. However, this contradicts $y_1(x) > y_2(x)$. ∎

Now, since our particular f is non-increasing in y for each fixed x, we can apply Proposition 1. Thus, solutions are unique.

Now consider the successive approximations. We have:

$$
\begin{aligned}
\phi_1(x) &= 0 + \int_0^x f(s, \phi_0) ds \\
&= \int_0^x f(s, 0) ds \\
&= \int_0^x (2s - \tfrac{4(0)}{s}) ds \\
&= \int_0^x 2s ds \\
&= s^2]_0^x = x^2 \\
\phi_2(x) &= 0 + \int_0^x f(s, \phi_1(s)) ds \\
&= \int_0^x f(s, s^2) ds \\
&= \int_0^x (2s - \tfrac{4s^2}{s}) ds \\
&= \int_0^x (-2s) ds \\
&= -s^2]_0^x = -x^2 \\
\phi_3(x) &= 0 + \int_0^x f(s, \phi_2(s)) ds \\
&= \int_0^x f(s, -s^2) ds \\
&= \int_0^x 2s ds \\
&= s^2]_0^x = x^2
\end{aligned}
$$

Thus, the successive approximations are $\phi_0(x) = 0, \phi_{2m-1}(x) = x^2, \phi_{2m}(x) = -x^2$ where $m = 1, 2, \cdots$. Therefore, the successive approximations do not converge because $\{\phi_m(x)\}$ has two cluster values for each $x \neq 0$. More precisely, the successive approximations oscillate. Thus, uniqueness of solutions does not imply convergence of successive approximations.

(In passing, someone may claim that since we have two constant subsequences, maybe one of the two constant subsequences is a solution. Let $\alpha(x) = x^2$. Then,

$$\alpha'(x) = 2x \neq -2x = 2x - \frac{4x^2}{x} = f(x, x^2) = f(x, \alpha(x)).$$

Thus, $\alpha(x) = x^2$ is not a solution. Now, let $\beta(x) = -x^2$. We then have

$$\beta'(x) = -2x \neq 2x = f(x, -x^2) = f(x, \beta(x))$$

Thus, $\beta(x) = -x^2$ is not a solution because $\beta'(x) \neq f(x, \beta(x))$.)

Does convergence of successive approximations imply uniqueness? The following is a counterexample to answer this question. Let us begin with

$$\frac{dy}{dx} = y^{\frac{1}{3}}, y(0) = 0. \tag{3.4}$$

Now,

$$\begin{aligned}
\phi_0(x) &\equiv 0 \\
\phi_1(x) &= y_0 + \textstyle\int_0^x f(s, \phi_0(s))ds \\
&= 0 + \textstyle\int_0^x f(s, 0)ds \\
&= 0 + \textstyle\int_0^x 0 ds \\
&= 0 \\
\phi_2(x) &= y_0 + \textstyle\int_0^x f(s, \phi_1(s))ds \\
&= 0 + \textstyle\int_0^x f(s, 0)ds \\
&= 0 + \textstyle\int_0^x 0 ds \\
&= 0
\end{aligned}$$

$\phi_n(x) = y_0 + \int_0^x f(s, \phi_{n-1}(s))ds = 0$. Hence, $\phi_n \to 0$, which is a solution to (3.4). Now, using separation of variables, we have that $y = (\frac{2x}{3})^{\frac{3}{2}}$ is a solution. Therefore, there is not uniqueness of solutions.

There are also nonuniqueness theorems. The following provides a sort of converse of Theorem 4.

Theorem 6 [5]:
Let $g(x, y)$ be continuous on $R_1 = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq 2b\}, g(x, 0) \equiv 0$ and $g(x, y) > 0$ for $y > 0$. Suppose that, for each $x_1 \in (0, a], u(x) \not\equiv 0$ is a solution of

$$\begin{aligned}
u'(x) &= g(x, u(x)), x \in [0, x_1] \\
u(0) &= u'_+(0) = 0.
\end{aligned}$$

Let $f \in C[R, \mathbf{R}]$, where $R = \{(x, y) : 0 \leq x \leq a, | y | \leq b\}$ and for $(x, w), (x, y) \in R$,

$$| f(x, w) - f(x, y) | \geq g(x, | w - y |).$$

Then,

$$\begin{aligned}
x' &= f(x, y), x \in [0, a] \\
x(0) &= 0
\end{aligned}$$

has at least two solutions.

A proof of Theorem 6 can be found in [5].

Here is another result concerning uniqueness.

Theorem 7 [13]:
Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and let $f(0) = 0$. If either of the following hold:

1. There exists an $\epsilon > 0$ such that $f(y) > 0$ on $(0, \epsilon]$ and $\int_0^\epsilon \frac{dw}{f(w)}$ exists, or

2. There exists an $\epsilon > 0$ such that $f(y) < 0$ on $[-\epsilon, 0)$ and $\int_{-\epsilon}^0 \frac{dw}{f(w)}$ exists,

then there exists a nonzero solution to $y' = f(y)$, $y(0) = 0$ (and hence solutions are not unique).

Proof:
Assume without loss of generality that 1) holds. Define $G : [0, \epsilon] \to \mathbf{R}$ by $G(z) = \int_0^z \frac{dw}{f(w)}$. Since $f(y) > 0$ for all $y \in (0, z]$, then $\frac{1}{f(y)} > 0$ for all $y \in (0, z]$. Thus, G is increasing and hence G^{-1} exists. Let $\overline{x} = G(\epsilon)$. Now define $v : [0, \overline{x}] \to \mathbf{R}$ by $v(x) = G^{-1}(x)$. Applying a well-known theoren of calculus (see, for example, Theorem 7.7 of [12]), we have for $x \in (0, \overline{x}]$

$$v'(x) = (g^{-1})'(x) = \frac{1}{G'(G^{-1}(x))} = \frac{1}{\frac{1}{f(v(x))}} = f(v(x)).$$

Also, $v(0) = 0$, so v is a nonzero solution. ∎

Related ideas can be found in [1].

Another result concerning uniqueness is the following.

Theorem 8:
Let $f : [0, \overline{x}] \times \mathbf{R} \to \mathbf{R}$ be continuous. Let $w_n \to w_\infty$. For each $n = 1, 2, \cdots, \infty$, assume that

$$\begin{aligned} y' &= f(x, y), x \in [0, \overline{x}] \\ y(0) &= w_n \end{aligned}$$

has a unique solution, denoted by y_n. Then, $\{y_n\}$ converges uniformly on $[0, \overline{x}]$ to y_∞. See, for example, [11].

We also note that examples exist for f defined on a rectangle R with the property that $y' = f(x, y), y(x_0) = y_0$ has nonuniqueness of solutions for every $(x_0, y_0) \in R$.

One such example was given in [6]. A simpler example can be found in [3]-however, even this example is quite involved.

We have surveyed results concerning uniqueness of solutions for ordinary differential equations. We began with the consideration of uniqueness from a geometric perspective. Next, we presented several classical theorems which specify conditions sufficient for uniqueness. Finally, we investigated other results concerning uniqueness, for example, the relationship between convergence of successive approximations and uniqueness.

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